Dual Dig Level I (2008) - Solutions

1. Solve for x and y, where x and y are real numbers, and $i=\sqrt{-1}$: $(2x+5y)+(3x-2y)i=19i$.

Solution:

We need to solve the system:

$$\begin{cases} 2x+5y=0 & \leftarrow\cdot(2) \\ 3x-2y=19 & \leftarrow\cdot(5) \end{cases}$$

Multiply both sides of the first equation by 2 and both sides of the second equation by 5.

$$\begin{cases} 4x+10y=0 \\ 15x-10y=95 \end{cases}$$

$$\begin{aligned} 19x &= 95 \\ x &= 5 \end{aligned}$$

From our first equation:

$$2x+5y=0 \text{ and } x=5 \Rightarrow$$

$$\begin{aligned} 2(5)+5y&=0 \\ 10+5y&=0 \\ 5y&=-10 \\ y&=-2 \end{aligned}$$

Answer: $x=5$ and $y=-2$

2. Simplify: $\sqrt{\frac{4}{3}}-\sqrt{\frac{3}{4}}$

Solution:

$$\begin{aligned} \sqrt{\frac{4}{3}}-\sqrt{\frac{3}{4}} &= \frac{2}{\sqrt{3}}-\frac{\sqrt{3}}{2} \\ &= \frac{4-3}{2\sqrt{3}} \\ &= \frac{1}{2\sqrt{3}}\cdot\frac{\sqrt{3}}{\sqrt{3}} \\ &= \frac{\sqrt{3}}{6} \end{aligned}$$

Answer: $\frac{\sqrt{3}}{6}$

3. Simplify: $\dfrac{\log 15}{\log\left(\dfrac{1}{15}\right)}$

Solution:

$$\frac{\log 15}{\log\left(\frac{1}{15}\right)} = \frac{\log 15}{\log 1 - \log 15}$$

$$= \frac{\log 15}{0 - \log 15}$$

Answer: -1

$$= \frac{\log 15}{-\log 15}$$

$$= -1$$

4. How many integers between 1 and 1000 are divisible by 3 but are not divisible by 4?

Solution:

Let $\lfloor \ \rfloor$ be the floor or "round down" operator.

There are $\left\lfloor \dfrac{1000}{3} \right\rfloor = 333$ integers from 1 through 1000 that are divisible by 3.

However, we need to delete those integers that are divisible by 12.
(If an integer is not divisible by 3, then it cannot be divisible by 12.)

We need to delete $\left\lfloor \dfrac{1000}{12} \right\rfloor = 83$ integers. We are left with: $333 - 83 = 250$ integers.

Answer: 250 integers.

5. Solve for a in terms of b and c if $\dfrac{1}{a} = \dfrac{1}{b} + \dfrac{1}{c}$. Assume that a, b, and c are nonzero.

Solution:

Multiply both sides by abc.

$$\frac{1}{a} = \frac{1}{b} + \frac{1}{c}$$

$$\frac{abc}{a} = \frac{abc}{b} + \frac{abc}{c}$$

$$bc = ac + ab$$

$$bc = a(c + b)$$

$$a = \frac{bc}{c+b} \text{ or } a = \frac{bc}{b+c}$$ (Answer)

6. Let $f(x)$ be a periodic function with a fundamental period of 5. If $f(1)=3$, $f(2)=4$, and $f(3)=5$, find the exact value of $f(72)$.

Solution:

Because 70 is a multiple of 5,

$$\begin{aligned} f(72) &= f(70+2) \\ &= f(2) \\ &= 4 \end{aligned}$$

Answer: 4

7. Find the equation of a quadratic function if its graph in the standard *xy*-plane has *x*-intercepts of 5 and -3 and a *y*-intercept of -8.

Solution:

$$y = a(x-5)(x+3)$$

Solve for *a*:

If $x=0$, then $y=-8$.

$$\begin{aligned} -8 &= a(0-5)(0+3) \\ -8 &= -15a \\ a &= \frac{8}{15} \end{aligned}$$

Answer:

$$y = \frac{8}{15}(x-5)(x+3), \text{ or}$$

$$y = \frac{8}{15}(x^2-2x-15), \text{ or}$$

$$y = \frac{8}{15}x^2 - \frac{16}{15}x - 8$$

8. A chemist is mixing solutions of acid together. How much pure water should be mixed with a 40% solution of acid to produce 20 liters of a 10% acid solution?

Solution:

Solution	Acid proportion	Quantity of solution	Quantity of acid
Pure water	0	x	0
40% acid	.4	$20-x$	$.4(20-x)$
10% acid	.1	20	2

Solve for x:

$$.4(20-x)=2$$

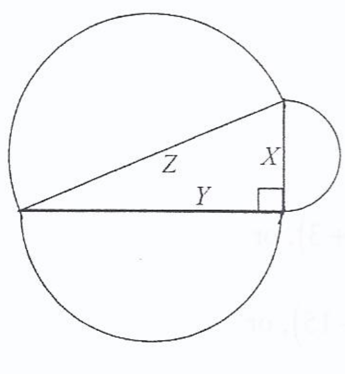
$$20-x=\frac{2}{.4}$$

$$20-x=5$$

$$x=15 \text{ liters}$$

Answer: 15 liters

9. The triangle shown is a right triangle and each side of the triangle is the diameter of a semicircle. For simplicity, we will refer to each semicircle by its diameter. If semicircle Z has an area of 169 m^2 and semicircle X has an area of 25 m^2, then what is the exact length of segment Y?



Solution:

For each side, express the square of the side length in terms of the area of the semicircle that has that side as its diameter. Let A be the area of the semicircle, let d be its diameter (and thus the length of the corresponding triangle side), and let r be its radius.

$$A=\frac{1}{2}\pi r^2$$

$$A=\frac{1}{2}\pi\left(\frac{1}{2}d\right)^2$$

$$A=\frac{\pi}{8}d^2$$

$$d^2=\frac{8A}{\pi}$$

Let d_X, d_Y, and d_Z be the lengths of triangle sides X, Y, and Z, respectively.

Apply the Pythagorean Theorem, and solve for d_Y.

$$(d_Z)^2 = (d_X)^2 + (d_Y)^2$$

$$\frac{8(169)}{\pi} = \frac{8(25)}{\pi} + (d_Y)^2$$

$$(d_Y)^2 = \frac{8(169)}{\pi} - \frac{8(25)}{\pi}$$

$$(d_Y)^2 = \frac{8}{\pi}(169 - 25)$$

$$(d_Y)^2 = \frac{8}{\pi}(144)$$

$$d_Y = \sqrt{\frac{8}{\pi}(144)}$$

$$d_Y = \sqrt{\frac{8}{\pi}} \cdot \sqrt{144}$$

$$d_Y = \frac{2\sqrt{2}}{\sqrt{\pi}} \cdot 12$$

$$d_Y = 24\sqrt{\frac{2}{\pi}} \text{ m}$$

Answer: $24\sqrt{\frac{2}{\pi}}$ m

10. A farmer raises only chickens and cows. There are 34 animals in all. One day, after falling on the ground, the farmer counts 110 legs total on all of his animals. How many chickens are there?

Solution:

Let x = the number of chickens.
Let y = the number of cows.

We need to solve the system for x:

$$\begin{cases} 2x + 4y = 110 & \leftarrow \cdot(-1/2) \\ x + y = 34 \end{cases}$$

Multiply both sides of the first equation by $-\frac{1}{2}$.

$$\begin{cases} -x-2y=-55 \\ x+\ \ y=34 \end{cases}$$

$$-y=-21$$

$$y=21$$

From our second equation:

$$x+y=34 \text{ and } y=21 \Rightarrow$$

$$x+(21)=34$$

$$x=13$$

Answer: $x=13$ chickens

11. Simplify $(1+i)^{16}$ completely, where $i=\sqrt{-1}$.

Solution:

$$\begin{aligned}(1+i)^{16} &= \left[(1+i)^2\right]^8 \\ &= \left[1+2i+i^2\right]^8 \\ &= \left[1+2i-1\right]^8 \\ &= \left[2i\right]^8 \\ &= (2)^8(i)^8 \\ &= 256\left(i^2\right)^4 \\ &= 256(-1)^4 \\ &= 256(1) \\ &= 256\end{aligned}$$

12. Solve: $x+6 \le 3x-4 \le x^2-8$. Write your answer in interval form.

Solution:

$$\begin{aligned} x+6 &\le 3x-4 \\ 10 &\le 2x \\ 5 &\le x \\ x &\ge 5 \end{aligned}$$

and

$$\begin{aligned} 3x-4 &\le x^2-8 \\ 0 &\le x^2-3x-4 \\ 0 &\le (x-4)(x+1) \\ (x-4)(x+1) &\ge 0 \end{aligned}$$

$x \ge 4$ or $x \le -1$ (by a sign chart, for example)

The set $\{x \mid x \ge 5\}$ is a subset of the set $\{x \mid x \ge 4 \text{ or } x \le -1\}$, so their intersection is the first set, $\{x \mid x \ge 5\}$.

Answer: $[5, \infty)$

13. A rubber ball is blasted straight up into the air. Disregarding air friction, the spin of the earth, etc., the height of the ball is given by: $h(t) = -16t^2 + vt + c$, where $h(t)$ represents the height of the ball (in feet from the ground) at time t (in minutes after the blast), and v represents the initial velocity, while c represents the initial height from which the ball is thrown. If the initial velocity is 64 feet/minute, and the ball is thrown from the top of a 176 ft tall building, at what time(s) [after the blast] is the height of the ball 96 feet?

Solution:

$$\begin{aligned} h(t) &= -16t^2 + 64t + 176 \\ 96 &= -16t^2 + 64t + 176 \\ 0 &= -16t^2 + 64t + 80 \\ 0 &= -16(t^2 - 4t - 5) \\ 0 &= -16(t-5)(t+1) \end{aligned}$$

The only nonnegative solution is: $t = 5$ minutes.
Answer: 5 minutes.

14. Let $f(x)=\dfrac{cx}{2x+3}$, where $x\neq-\dfrac{3}{2}$. Find all real values of c, if any, for which $f(f(x))=x$.

Solution:

$$f(f(x))=x$$

$$f\left(\frac{cx}{2x+3}\right)=x$$

$$\frac{c\left(\frac{cx}{2x+3}\right)}{2\left(\frac{cx}{2x+3}\right)+3}=x$$

$$\frac{\frac{c^2x}{2x+3}}{\frac{2cx+3(2x+3)}{2x+3}}=x$$

$$\frac{c^2x}{2cx+3(2x+3)}=x$$

$$\frac{c^2x}{2cx+6x+9}=x$$

$$c^2x=2cx^2+6x^2+9x$$

$$0=2cx^2+6x^2-c^2x+9x$$

$$0=\left(2cx^2+6x^2\right)+\left(9x-c^2x\right)$$

$$0=(2c+6)x^2+\left(9-c^2\right)x$$

We require both:

$$2c+6=0 \qquad\qquad 9-c^2=0$$

$$c=-3 \qquad \text{and} \qquad c^2=9$$

$$c=\pm3$$

Only $c=-3$ works.

Answer: -3

15. Al and Betty each pick a positive integer. The least common multiple of their integers is $2^3 \cdot 3^4 \cdot 5 \cdot 7$. The greatest common divisor of their integers is $2 \cdot 3 \cdot 5$. We find out that Al picked 210. What is Betty's integer?

Solution:

Observe that $210 = 2 \cdot 3 \cdot 5 \cdot 7$. Let's call Betty's integer "x".
Because their GCD is $2 \cdot 3 \cdot 5$, x must have 2, 3, and 5 as factors, but not 7.
Because their LCM is $2^3 \cdot 3^4 \cdot 5 \cdot 7$, x has no other prime factors, and x must have 2^3, 3^4, and 5 as the highest powers of 2, 3, and 5 in its prime-power factorization, respectively.

Therefore,

$$\begin{aligned} x &= 2^3 \cdot 3^4 \cdot 5 \\ &= 8 \cdot 81 \cdot 5 \\ &= 3240 \end{aligned}$$

Answer: 3240

16. Find all real solutions of: $6^{1+x} + 6^{1-x} = 37$.

Solution:

$$6^{1+x} + 6^{1-x} = 37$$

$$6 \cdot 6^x + \frac{6}{6^x} = 37$$

$$\frac{6 \cdot 6^{2x} + 6}{6^x} = 37$$

$$6 \cdot 6^{2x} + 6 = 37 \cdot 6^x$$

$$6 \cdot 6^{2x} - 37 \cdot 6^x + 6 = 0$$

Let $u = 6^x \Rightarrow u^2 = 6^{2x}$.

$$6u^2 - 37u + 6 = 0$$

$$(6u-1)(u-6) = 0$$

$$\begin{aligned} 6u - 1 &= 0 \\ u &= \frac{1}{6} \\ 6^x &= \frac{1}{6} \\ x &= -1 \end{aligned} \quad \text{or} \quad \begin{aligned} u - 6 &= 0 \\ u &= 6 \\ 6^x &= 6 \\ x &= 1 \end{aligned}$$

Answer: $\{-1, 1\}$

17. In how many ways can $10 be changed into just dimes and quarters, provided that at least one of each coin must be used?

Solution:

The total value of the quarters can only be taken in 50-cent increments, from 50 cents through $9.50; remember that at least one dime and at least one quarter must be used. There are 19 possible mixtures.

Alternate Solution:

If d is the number of dimes used, and if q is the number of quarters used, then (in cents):

$$10d + 25q = 1000$$
$$2d + 5q = 200$$
$$2d = 200 - 5q$$
$$2d = 5(40 - q)$$

Both sides must be even and positive, so q must be even; furthermore:

$$40 - q > 0$$
$$40 > q$$
$$q < 40$$

There are 19 positive even integer values for q between 1 and 39.
Answer: 19 ways

18. The following five numbers are placed into a jar and shaken: 71, 76, 80, 82, and 91. The numbers are drawn out, one by one, in such a way that the average of the first two numbers drawn is an integer, the average of the first three numbers drawn is an integer, and the average of the first four numbers drawn is an integer. What must be the third, fourth, and fifth numbers drawn? Hint: Consider the "three numbers drawn" case first by using remainders when the numbers are divided by 3.

Solution:

In number theory, we would use (mod n) arithmetic, in which we study remainders when integers are divided by n. Observe that the average of n numbers is an integer if and only if their sum is a multiple of n.

If the integers 71, 76, 80, 82, and 91 are divided by 3, the remainders (think: mod 3 arithmetic) are: 2, 1, 2, 1, and 1. The first three numbers drawn must have a sum that is a multiple of 3, so the remainders of those numbers must add up to a multiple of 3. The first three numbers drawn must then be 76, 82, and 91 in some order, since they are the only three numbers with this property.

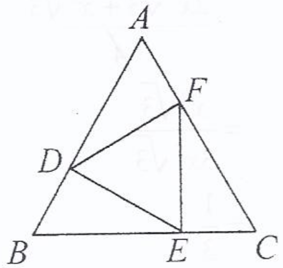
The first two numbers drawn (among 76, 82, and 91) must have a sum that is even, so they must be the two even numbers, 76 and 82, in either order. The third number must then be 91.

The sum of the first three numbers drawn is $76 + 82 + 91 = 249$, which is odd. In order for the sum of the first four numbers to be divisible by 4, the fourth number must be odd. Since 71 is odd, and 80 is not, the fourth number must be 71; observe that $76 + 82 + 91 + 71 = 320$, which is a multiple of 4. The fifth number must then be 80.

Answer:

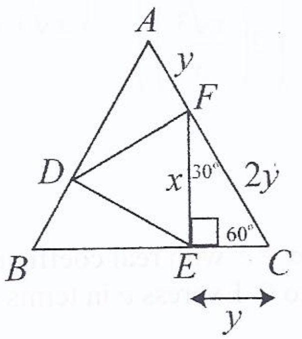
(First and second numbers: 76 and 82, in either order)
Third: 91
Fourth: 71
Fifth: 80

19. In the figure below, equilateral triangle *DEF* is inscribed in equilateral triangle *ABC*. We have that $\overline{EF} \perp \overline{BC}$. What is the ratio of the area of triangle *DEF* to the area of triangle *ABC*? Write your answer in simplest form.



Solution:

Because triangle *DEF* is equilateral, and because each side faces the 60° angle of a $30^\circ - 60^\circ - 90^\circ$ triangle (see why for yourself), each of the three triangles *AFD*, *BDE*, and *CEF* are congruent.



Let *x* be the length of $\overline{EF}$. Then, the length of $\overline{EC}$ is given by: $y = \dfrac{x}{\sqrt{3}} = \dfrac{x\sqrt{3}}{3}$.

The area of triangle *CEF* is given by: $\dfrac{1}{2}\left(\dfrac{x\sqrt{3}}{3}\right)(x) = \dfrac{x^2\sqrt{3}}{6}$.

The combined area of the three triangles is: $\dfrac{3x^2\sqrt{3}}{6} = \dfrac{x^2\sqrt{3}}{2}$.

The area of triangle *DEF* is given by:

$\dfrac{1}{2}(x)\left(\dfrac{x\sqrt{3}}{2}\right) = \dfrac{x^2\sqrt{3}}{4}$.

The ratio of the area of triangle DEF to the area of triangle ABC is given by:

$$\frac{\text{Area of triangle } DEF}{\text{Area of triangle } ABC} = \frac{\frac{x^2\sqrt{3}}{4}}{\frac{x^2\sqrt{3}}{2}+\frac{x^2\sqrt{3}}{4}} = \frac{\frac{x^2\sqrt{3}}{\not{4}}}{\frac{2x^2\sqrt{3}+x^2\sqrt{3}}{\not{4}}} = \frac{x^2\sqrt{3}}{3x^2\sqrt{3}} = \frac{1}{3}$$

More simply: The ratio between areas of similar triangles may be given by the square of the ratio between corresponding side lengths. For example:

$$\left(\frac{EF}{AC}\right)^2 = \left(\frac{x}{3y}\right)^2 = \left(\frac{x}{3\left(\frac{x\sqrt{3}}{3}\right)}\right)^2 = \left(\frac{x}{x\sqrt{3}}\right)^2 = \left(\frac{1}{\sqrt{3}}\right)^2 = \frac{1}{3}.$$

Answer: $\frac{1}{3}$

20. Consider the polynomial x^3+ax^2+bx+c with real coefficients, where all three roots (or zeros) are real, and two of the roots have sum equal to 0. Express c in terms of a and b.

Solution:

Let the roots be r_1, $-r_1$, and r_2.

$$x^3+ax^2+bx+c = d(x-r_1)(x-(-r_1))(x-r_2)$$

where d is 1, because the leading coefficient on the given polynomial form is 1

$$= (x-r_1)(x+r_1)(x-r_2)$$
$$= (x^2-r_1^2)(x-r_2)$$
$$= x^3 - r_2x^2 - r_1^2x + r_1^2r_2$$

By matching coefficients, we have: $\begin{cases} c = r_1^2r_2 \\ a = -r_2 \\ b = -r_1^2 \end{cases} \Rightarrow c = (-b)(-a) = ab$

Answer: $c = ab$.